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# Relationship between Coast Arc Length and Switching Function Value during Optimization


W.T. Fowler*

*University of Texas at Austin, Austin, Texas*

and

P.M. O'Neill†

*McDonnel Douglas Aircraft Co., St. Louis, Mo.*


## Introduction

In the determination of optimal low-thrust interplanetary trajectories containing a coast arc, several procedures have been developed for handling the coast arc. These procedures usually involve either 1) the guessing of and iteration on the coast arc entry and exit times, $t_1$ and $t_2$ respectively, or 2) the use of a switching function, $\Gamma(t)$, to determine $t_1$ and $t_2$. Most procedures based on iteration on $t_1$ and $t_2$ do not use switching function information while most procedures based on the switching function do not employ iteration on $t_1$ and $t_2$.

The procedure described uses both the switching function and information equivalent to $t_1$ and $t_2$ (the length, $T_0$, and center, $t_c$, of the coast arc are used). The problem is formulated as a two-point boundary-value problem and guesses are made for all unknown initial states and Lagrange multipliers. Then initial guesses are made for $t_c$ and $T_0$.

A standard perturbation-type optimization procedure (Refs. 1-5) is used to find the unknown initial states and multipliers with $T_1$ held constant. During each iteration, $\Gamma(t)$ is monitored, but it does not control the coast arc. Both $T_0$ and $t_c$ are held constant until the unknown initial conditions have been converged. Then, $t_c$ is shifted so that the values of the switching function $\Gamma(t_1)$ and $\Gamma(t_2)$ become equal. This equalization or balancing procedure usually takes only two or three iterations to reconverge the trajectory and produces a value of the switching function which will be called $\Gamma_{\mathrm{BAL}}$ ($\Gamma_{\mathrm{BALANCED}}$). Figure 1 shows a typical switching function just prior to balancing. The switching function for the reconverged "balanced" trajectory will differ somewhat from the switching function prior to balancing, but usually the differences are small. After two different values of $\Gamma_{\mathrm{BAL}}$ have been obtained corresponding to two differing values of $T_0$, a linear extrapolation can be made to determine the optimal value of $T_0$. The fact that the $\Gamma_{\mathrm{BAL}}$ vs $T_0$ graph is linear was discovered during a series of optimization runs. It had not been planned to use the $\Gamma_{\mathrm{BAL}}$ vs $T_0$ plot to predict the optimal coast arc length, but when the linearity of the plot was discovered, it was used.

An outline of the procedure is now presented. 1) Guess all unknown initial states and multipliers. 2) Converge an optimal trajectory containing no coast arc. However, monitor the switching function, $\Gamma(t)$, during this process. 3) If $\Gamma(t) > 0$ for the entire converged trajectory, no coast arc is required and the problem is solved. 4) If $\Gamma(t) < 0$ for some finite interval on the converged trajectory, choose values for $T_0$ and $t_c$ [use the $\Gamma(t)$ plot as an aid]. 5) Converge an optimal trajectory containing the *specified* coast arc (i.e., fixed $T_0$ and $t_c$) using the converged values from step 2 as initial guesses for the unknown states and multipliers. Again monitor $\Gamma(t)$. 6) Shift the value of $t_c$ until $|\Gamma(t_1) - \Gamma(t_2)| < \epsilon$ (a small number) Note that each shift of $t_c$ will require reiteration of the optimization process (usually only one or two iterates are required). 7) Once a point on the $\Gamma_{\mathrm{BAL}}$ vs $T_0$ plot has been determined, change $T_0$ by a small amount. The relation

$$\delta t_i = \frac{-\alpha \Gamma_{\mathrm{BAL}}(t_0)}{\Gamma(t_i)} \quad (i=1,2)$$

where $0 < \alpha < 1$, has been found to work well. 8) Repeat steps 5 and 6 using the converged initial states and multipliers from the last pass (step 6) as initial guesses. This will result in a second point on the $\Gamma_{\mathrm{BAL}}$ vs $T_0$ plot. 9) Extrapolate linearly to find the value of $T_0$ at which $\Gamma_{\mathrm{BAL}}(T_0) = 0$. 10) Using the most recent initial conditions, the correct $T_0$, and the current value of $t_c$, repeat steps 5 and 6 to obtain the optimal trajectory. The result should give $\Gamma(t_1) = \Gamma(t_2) = 0$ upon reconvergence.

## Numerical Example

The example problem studied was a low-thrust Earth-Jupiter transfer with one coast arc allowed but not required in the trajectory. Only the heliocentric phase of the mission was considered and only the solar gravitation was modeled (no drag, radiation pressure, etc.) All trajectories were assumed to leave Earth on the same day (November 15, 1983). The thrust level was full thrust or zero (no throttling).

The equations of motion for the vehicle are given by

$$\dot{x}_1 = -\frac{\mu}{r^3} x_4 + \frac{u_3 c}{x_7} \cos u_1 \cos u_2 \tag{1a}$$

$$\dot{x}_2 = -\frac{\mu}{r^3} x_5 + \frac{u_3 c}{x_7} \cos u_1 \sin u_2 \tag{1b}$$

$$\dot{x}_3 = -\frac{\mu}{r^3} x_6 + \frac{u_3 c}{x_7} \sin u_1 \tag{1c}$$

$$\dot{x}_4 = x_1 \qquad \dot{x}_5 = x_2 \tag{1d}$$

$$\dot{x}_6 = x_3 \qquad \dot{x}_7 = -u_3 \tag{1e}$$

where $\mu$ is the product of the universal gravitational constant, $G$, and the Sun's mass, $r$ is the distance of the vehicle from the Sun, $c$ is exhaust velocity of the low-thrust engine relative to the vehicle, $x_1$, $x_2$, $x_3$ are the vehicle's velocity components in


Presented at the ASS/AIAA Astrodynamics Specialist Conference, Nassau, Bahamas, July 28-30, 1975; submitted Nov. 5, 1975; revison received Feb. 24, 1976.

Index category: Lunar and Interplanetary Trajectories.


*Associate Professor. Member AIAA.

†Engineer.

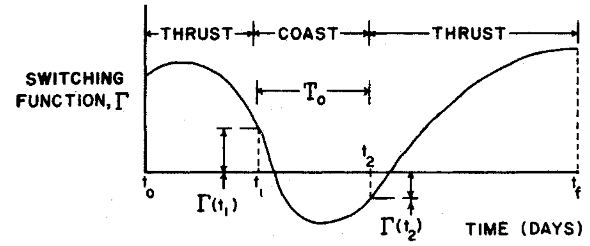


**Fig. 1 Typical switching function prior to balancing.**

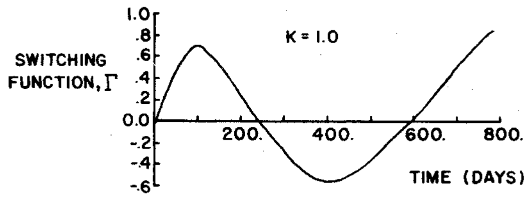


**Fig. 2 $\Gamma_{BAL}$ vs $T_0$ plot used to find optimal coast arc length.**

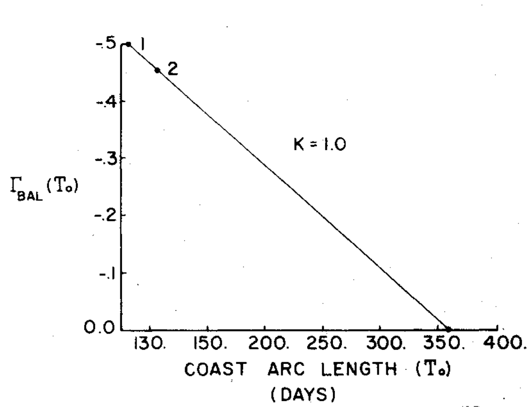


**Fig. 3 Converged switching function.**

heliocentric nonrotating rectangular Cartesian coordinates, $x_4$, $x_5$, $x_6$ are the vehicle's position components, $u_3$ is the absolute value of the mass flow rate of the engine with $u_3 = 0$ or $\beta$ and $u_1 u_2$, are the control angles which define the thrust direction.

The performance index used was a linear combination of flight time and propellant mass expended. Previous experience with low thrust Earth-Jupiter trajectories has shown that a performance index based entirely on propellant mass often leads to very large flight times if coast arcs are allowed. Trajectories which minimize flight time result in no coast arcs. The performance index used is given by

$$J = -x_7(t_f) + K\beta t_f \tag{2}$$

The necessary conditions for minimization of the performance index subject to the equations of motion and the initial and terminal constraints are given by

$$\dot{\bar{x}} = \frac{\partial H^T}{\partial \lambda}, \; \dot{\bar{\lambda}} = \frac{-\partial H^T}{\partial \bar{x}} = 0, \; \frac{\partial H^T}{\partial \bar{u}} = 0 \tag{3a}$$

$$\bar{\lambda}(t_f) = \frac{\partial R^T}{\partial \bar{x}(t_f)} = \left\{ \frac{\bar{v}}{-1} \right\} \tag{3b}$$

$$\bar{M}(t_f) = \bar{0} \quad \left(H + \frac{\partial R}{\partial t_f}\right)\Big|_{t_f} = 0 \tag{3c}$$

where the time of flight, $t_f$, is unspecified while $t_0$ is fixed. The variational Hamiltonian, $H$, is defined in the standard manner, and $R$ is given by

$$R = -x_7(t_f) + K\beta t_f - v^{-T}\bar{M}$$

where $\bar{M}$ is the vector of terminal constraints and $\bar{v}$ is a vector of unknown constant multipliers. The $\bar{\lambda}$ vector is the usual time dependent Lagrange multiplier vector.

Eliminating the control in the usual fashion, using Eq. (3c) plus the requirement that $\partial^2 H/\partial \bar{u}^2$ be positive semi-definite, we get

$$\cos u_1 = \delta/\Delta \qquad \sin u_1 = -\lambda_3/\Delta$$
$$\cos u_2 = -\lambda_1/\delta \qquad \sin u_2 = -\lambda_2/\delta$$
$$u_3 = \left\{ \begin{array}{l} \beta \text{ for engine on} \\ 0 \text{ for engine off} \end{array} \right\} \tag{4c}$$

where

$$\delta = (\lambda^2 + \lambda_2^2)^{1/2}, \; \Delta = (\lambda_2^2 + \lambda_3^2)^{1/2} \text{ and } \Gamma = \lambda_7 + \frac{c\Delta}{x_7} \tag{5}$$

As was stated earlier, the switching function $\Gamma$ defined above was not used to determine coast arc entry and exit times during the iteration process, but it was used in the determination of the optimal coast arc length.

Optimal trajectories were obtained for several values of the weighting parameter $K$, in Eq. (2). One trajectory, with $K = 15.0$, exhibited no coast arc. The $\Gamma_{BAL}$ vs $T_0$ curve for the case $K = 1.0$ is shown in Fig. 2. Note that the two values chosen for $T_0$ were around 140 and 160 days while the optimal coast arc was about 350 days. The switching function for the $K = 1.0$ case is shown in Fig. 3.

## Conclusion

An unexpected relation between coast arc length and switching function value during optimization has been found. A balancing procedure used on the switching function led to the discovery of a linear relation between the guessed coast arc length and the balanced switching function value. Through extrapolation, the optimal value of the coast arc can be precisely determined. The result is limited at present to cases where the number of switchings is two or fewer, but extensions may be possible.

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# Roll-Rate Stabilization of a Missile Configuration with Wrap-Around Fins

Peter Daniels* and Samuel R. Hardy†
*Naval Surface Weapons Center, Dahlgren, Va.*

## Nomenclature

$d$ = reference diameter, 1.6 in.
$p$ = steady-state roll rate, rad/sec

Received November 17, 1975; revision received December 22, 1975.
Index categories: LV/M Aerodynamics; LV/M Configurational Design; LV/M Dynamics, Uncontrolled.

*Staff Engineer.
†Aerospace Engineer.